GRAPHICAL REPRESENTATION IN ARITHMETIC AND ALGEBRA.

WHEN an experience is strange to us, we try to find points of similarity with more familiar experiences; and we call the new facts by the names, and represent them by the ideas, of the old facts with which we are already well acquainted. Such, in general, are all scientific theories: they systematize our new knowledge, and suggest advances. Similar advantages are derived from geometric representation of numerical relations. By this means ideas otherwise abstruse are endowed with the reality that belongs to parallel ideas in visible form. When we consider that the whole structure of modern mathematics, even in its most abstruse branches, is largely dependent upon geometric representation, it would be strange if teachers should hesitate to use diagrams freely for representing the ideas of elementary arithmetic and algebra.

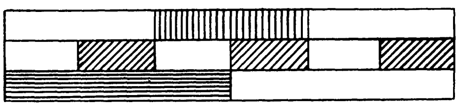


FIG. 1.—Reduction to a Common Denominator.

The most important use of geometric representation in arithmetic is for multiplication. The product-rectangle, divided into strips of unit squares, is found in most text-books of arithmetic—the multiplicand being the number of squares in a strip, and the multiplier the number of strips. It is generally used, however, only for showing that a rectangular area is numerically equal to the product of its dimensions; and generally only integral dimensions are considered.

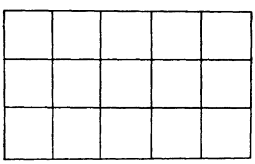


FIG. 2.—Multiplication of 5 by 3.

Even if we restrict ourselves to this simple diagram, and to integral factors, there are one or two important inferences to be drawn. To begin with, we may count our strips either vertically or horizontally; the height is the multiplicand in the first case, the base in the second; and the conclusion that a product is independent of the order of its

factors may be based upon something more real than a multiplication table. Again, we have here a test for prime numbers, as numbers that cannot be represented by a rectangle of squares. In applying

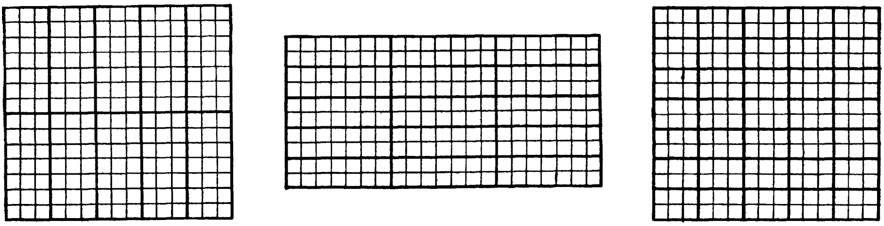


FIG. 3.—$2\times3\times7\times5=2\times7\times5\times3=7\times3\times2\times5$.

this test the pupil can easily be made to see that it is not necessary to try factors larger than the square root of the number he is trying to factor. The name "square root" need not be used—only the

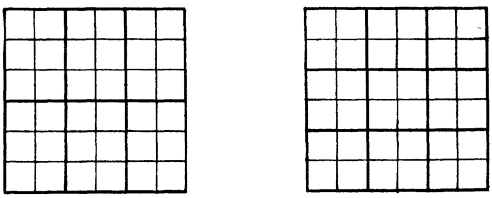


FIG. 4.—$3\times2\times2\times3=2^2\times3^2$.

trials need not be continued after the progressively increasing multiplier becomes greater than the decreasing multiplicand.

Continuous multiplication of several multipliers can be represented

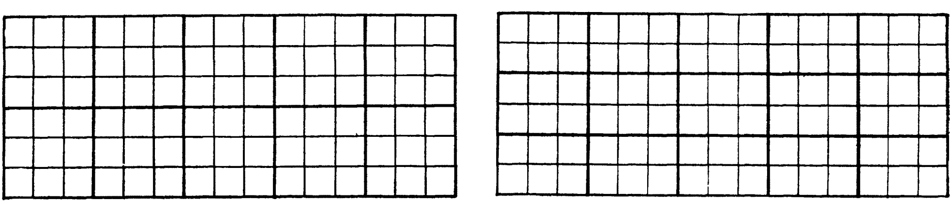


FIG. 5.—$2\times3\times3\times5=3^2\times2\times5$.

by an obvious extension of this diagram: a product of three factors by a row of rectangles, a product of four factors by a larger rectangle made up of equal rows of smaller rectangles, a product of five factors by a row of these larger rectangles, and so on. The commutative and associative laws can be illustrated by such diagrams.

When we come to the multiplication of fractions, our diagrams give us the best of definitions. Of course, the possibility of representing fractions is limited, but for halves, thirds, fifths, and tenths we may safely depend upon the pupil and his decimally divided cross-section paper.

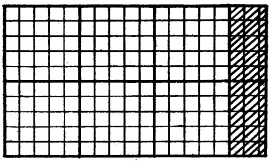


FIG. 6.—$2\times 3\frac{1}{2}$.

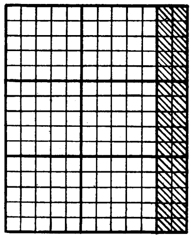


FIG. 7.—$2\frac{2}{5}\times 3$.

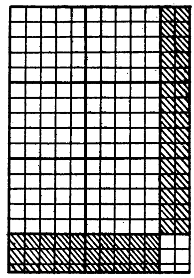
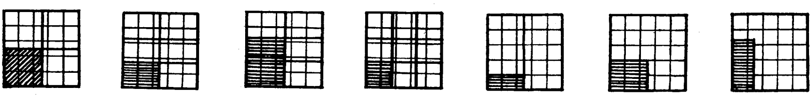


FIG. 8.—$2\frac{2}{5}\times 3\frac{1}{2}$.



$\frac{1}{2}\times\frac{1}{2}$. $\frac{1}{3}\times\frac{1}{2}$. $\frac{2}{3}\times\frac{1}{2}$. $\frac{1}{3}\times\frac{1}{3}$. $\frac{1}{5}\times\frac{1}{2}$. $\frac{2}{5}\times\frac{1}{2}$. $\frac{7}{10}\times\frac{3}{10}$.

FIG. 9.

The shaded areas in Figs. 6–8 inclusive show the products of the fractional part of one factor and the integral part of the other. Diagrams like those in Fig. 9 should be used to show that the product of two proper fractions is numerically less than either.

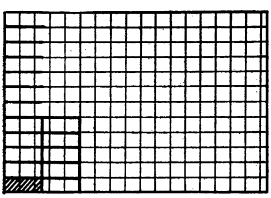


FIG. 10.—$2\frac{2}{5}\times 3\frac{1}{2}=\frac{1}{5}\times\frac{1}{2}\times 12\times 7=\frac{1}{10}\times 12\times 7$.

In Fig. 10, the multiplication $2\frac{2}{5}\times 3\frac{1}{2}$ becomes $\frac{12}{5}\times\frac{7}{2}$; that is, taking the new fractional units, each of which is a rectangle, not a square, the base of the product-rectangle extends along a strip of seven of these units, and the altitude extends across twelve of these strips.

The practical application of this discussion is the carpet problem; and in younger classes the rectangles should habitually be plans of floors, or chalked-out areas on school pavements or platforms. Their unit squares would be square yards of carpet, specified as straw matting or ingrain, or whatever else comes in yard widths; or linoleum or pine boards, which are sold by the square foot.

In later work the brussels carpet, coming in three-quarter yard widths, gives a thorough-going illustration of the fractional divisor.

Pupils can measure and plat their floors, halls, and stairs at home, and draw a "scale of breadths" on tracing-paper, which may be laid over the floor plans to give by inspection the number of breadths—a check on their numerical calculation. If the floor plans are drawn on the ordinary scale, $\frac{1}{4}$ inch to one foot, the scale of breadths will be a set of parallel lines $\frac{3}{16}$ of an inch apart.

The practical restriction that, whatever the shape of the plan, the breadths must be bought with square ends is by no means worthless as mathematical discipline; neither is the restriction of matching patterns.

Besides this practical interest, the product-rectangle is of very great theoretical importance. Fig. 8, for example, serves to illustrate the binomial product $(x+a)$ $(y+b)$, where $x=2$, $a=\frac{2}{5}$, $y=3$, $b=\frac{1}{2}$. An obvious extension of this representation is for the case where a and b are negative. Thus, in Fig. 11, when AB represents x, RB a, BC y, BR is measured back on BA, and the rectangle $BCQR$ is—ay, $ARQD$ being $(x-a)$ y and also $xy-ay$.

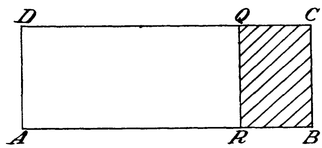


FIG. 11.—y $(x-a)$.

The product $(x-a)$ $(y-b)$ requires a further complication of the diagram, and completes a set of diagrams which may be used in teaching the law of signs in multiplication. They do not, it is true, furnish an explanation of that law, and are not even of use as a mnemonic device; but they do furnish a parallel set of geometric facts with which the algebraic ideas can be associated.

When the study of formal geometry is begun the idea of the product-rectangle can be utilized, instead of rejected, in the proofs of the correspondence between areas and products, and the extension of these proofs to incommensurables.

Next after the product-rectangle in importance, and somewhat later in the pupils' development, comes the plane of analytic geometry, utilized for the representation of equations with two letters. Experience has abundantly shown that the diagrams described as follows have a very illuminating effect on pupils beginning elimination.

In beginning the study of simultaneous equations, the first thing to show is that an equation of condition with two unknown letters has

an unlimited number of solutions. A list of some of these solutions can be made by taking a succession of supposed values of one letter and finding the corresponding values of the other letter. Each pair of corresponding values forms one "answer" to the equation, and any diagram used to represent this answer must represent a value of x and at the same time a value of y. Such a diagram is formed by taking two algebraic scales and setting them perpendicular to each other with their zero points together. Call the horizontal line the x-scale,

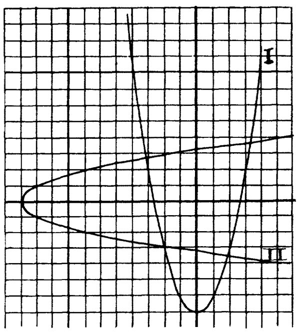


FIG. 12.—Diagram for Equations $x^2+y=7$; $x+y^2=11$.

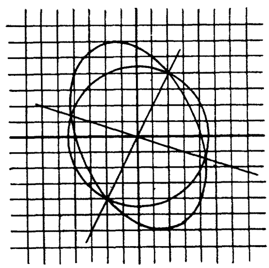


FIG. 13.—Diagram for Equations $x^2+y^2=20$; $bx^2+xy+y^2=32$.

or axis of x, and the vertical line the y-scale, or axis of y. Then imagine vertical lines drawn through all the points of the axis of x, and horizontal lines through all the points of the axis of y. A point representing any particular answer to this equation would be on the same vertical line as the value of x in the x-scale, and on the same horizontal line as the value of y in the y-scale.

Taking a partial list of the solutions of any equation of two letters, and marking the points which represent those answers, the pupil will recognize a shape in the diagram; can prophesy the location of a new answer, and verify his prophecy; and can be led to believe, although it may be beyond him to prove with rigor, that the completed shape indicated by his separate points furnishes a complete list of answers, an infinite list of answers, to one equation with two unknown letters; and that the accuracy of the answers obtained by him from that list depends only on his accuracy in drawing and measurement.

Elimination should be taught first for linear equations, plotting

each pair in a separate diagram, and pointing out that the answers give the co-ordinates of the intersection, as an answer which appears in both lists. After this fact has become familiar by the working out of several examples, and after the pupil has invented, upon slight suggestion, the plan of plotting only two points for each line, he should be cautioned that there are two-lettered equations that have curves for their lists of answers, and should plot two or three, substituting

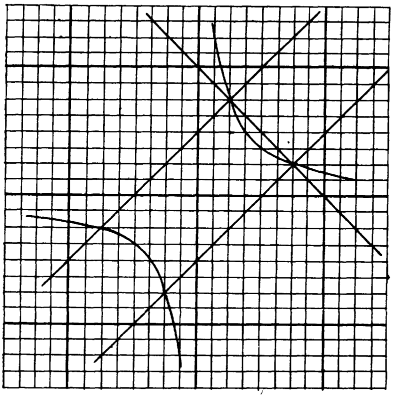


FIG. 14.—Diagram for $xy=12$; $x+y=8$.

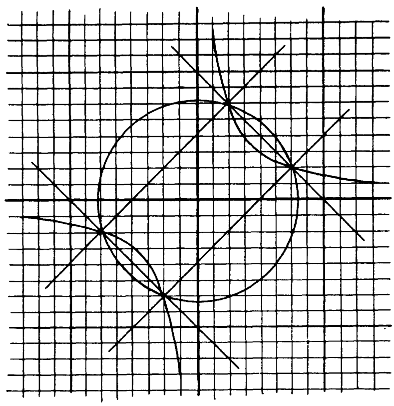


FIG. 15.—Diagram for $x^2+y^2=40$; $xy=12$.

successive integral values of one letter and solving the quadratic for the other. In this way he gets a diagram for the two solutions of a linear-quadratic pair, and for the four solutions of a pair of quadratics which he may be unable to solve algebraically. Thus for the equations $x^2+y=7$ and $x+y^2=11$, he gets a pair of parabolas intersecting in four points; and he can be taught to appreciate the argument that, since there are four answers, the equation that he gets by substitution will have four factors—a number which he cannot generally separate.

Another familiar type of simultaneous quadratics furnishes for a diagram a pair of central conics.

By combining these equations so as to eliminate the numerical terms we obtain the pair of straight lines $(2x-y)\ (x+3y)=0$, which pass through the intersections of the two given curves; and two of the answers are obtained by combining one of these linear equations with either of the given equations.

These diagrams show very clearly the way in which the different values of x and y belong together; and between what equations elimination can properly take place.

The method usually employed for eliminating when the simultaneous quadratics are symmetrical has quite an interesting diagram. The straight lines obtained in this method are pairs of parallels; and when both equations are quadratic the lines form a rectangle of which the corners are the answers.

GEO. W. EVANS.

THE LATIN SCHOOL,
Boston, Mass.